PLANE REAL ALGEBRAIC CURVES OF ODD DEGREE WITH A DEEP NEST

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Abstract. We apply the Murasugi-Tristram inequality to real algebraic curves of odd degree in $\mathbf{RP}^2$ with a deep nest, i.e. a nest of the depth $k-1$ where $2k+1$ is the degree. For such curves, the ingredients of the Murasugi-Tristram inequality can be computed (or estimated) inductively using the computations for iterated torus links due to Eisenbud and Neumann as the base case of the induction and Conway's skein relation as the induction step.

As an example of applications, we prove that some isotopy types are not realizable by M-curves of degree 9.

In Appendix B, we give some generalization of the skein relation.

1. Introduction and statement of the results

In this paper we apply the Murasugi-Tristram inequality (as in [7–9]) to study real algebraic curves of odd degree in $\mathbf{RP}^2$ with a deep nest, i.e. curves which have a nest of the depth $k-1$ where $2k+1$ is the degree. We study also analogous curves on real smooth ruled surfaces (curves satisfying conditions (1)–(4) below). For curves with a deep nest, the braid defined in [7] is uniquely determined by the arrangement of the curve with respect to the pencil of lines centered at a point inside the nest (the arrangement with respect to the fibers on the ruled surfaces). Moreover, if the degree is odd then the right hand side of the Murasugi-Tristram inequality (the signature and the nullity of the braid) can be computed inductively in some cases (and estimated in the other cases). The base case of the induction uses the computations for iterated torus links due to Eisenbud and Neumann, and the induction step is Conway's skein relation. As an example of applications, we prove that some isotopy types are not realizable by M-curves of degree 9.

In Appendix B, we give some generalization of the skein relation.

Let us fix an integer $n \geq 1$. Let $\pi_n : \Sigma_n \to \mathbf{P}^1$ be the fiberwise compactification of the line bundle $\mathcal{O}(n)$. Let E_n be the infinite section (i.e. $E_n^2 = -n$). If $n=1$ then Σ_n is the blown up $\mathbf{P}^2$. Otherwise Σ_n is a minimal rational smooth ruled surface — Hirzebruch surface. If $n \geq 1$ then Σ_n has a unique real form (Σ_0 has two real forms: hyperboloid and ellipsoid but we shall not consider the case $n=0$). For a real algebraic variety X, we denote the set of its real (resp. complex) points by $\mathbf{R}X$ (resp. by X).

We have a fibration $\pi_n : \mathbf{R}\Sigma_n \to \mathbf{RP}^1$ with the fiber $\mathbf{RP}^1$. If n is even then $\mathbf{R}\Sigma_n$ is a torus; if n is odd then $\mathbf{R}\Sigma_n$ is a Klein bottle.

For an algebraic curve A in Σ_n, we define the *bidegree of* A as $(A\cdot F, A\cdot E_n)$ where F is a fiber of π_n. The curves of a given bidegree (m,l) form a linear system on Σ_n. If (x,y) is a coordinate system in $\Sigma_n \setminus (F \cup E_n)$ such that the fibers are given by

$\{x = \text{const}\}$ then the Newton polygon of a generic curve of bidegree (m,l) is $(0,0)$-$(l+mn,0)$-(l,m)-$(0,m)$. The genus of such a curve is $g_{m,l} = (m-1)(mn/2+l-1)$ (the number of interior integral points in the Newton polygon).

In this paper we study real algebraic curves A on Σ_n satisfying the following conditions:

(1) A is non-degenerate;
(2) the bidegree of A is $(m,0)$ where $m = 2k+1$ is odd;
(3) There exists $A_0 \subset \mathbf{R}A$ such that $\pi_n|_{A_0}$ is a covering of $\mathbf{RP}^1$ of degree $m-2$;
(4) The curve A is not hyperbolic, i.e. for any connected component V of $\mathbf{R}A \setminus A_0$, the topological degree of the mapping $\pi_n|_V : V \to \mathbf{RP}^1$ is zero.

Example. If $n = 1$ then Σ_1 is $\mathbf{P}^2$ blown-up at a real point p and A is the strict transform of a smooth real curve of degree $m = 2k+1$ with a nest of ovals $O_1, \dots, O_{k-1}$, such that $p \in \operatorname{Int} O_1 \subset \dots \subset \operatorname{Int} O_{k-1}$.

Suppose that a curve A on Σ_n satisfies the conditions (1)–(4). If n is odd then A_0 is a disjoint union of k circles $A_0 = \mathcal{J} \sqcup O_1 \sqcup O_2 \sqcup \dots \sqcup O_{k-1}$, moreover, $\pi_n|_{\mathcal{J}} : \mathcal{J} \to \mathbf{RP}^1$ is a diffeomorphism and $\pi_n|_{O_i} : O_i \to \mathbf{RP}^1$ is a double covering. We call $\mathcal{J}$ *the odd branch of* A. If n is even then A_0 is a disjoint union of $2k$ circles, each being mapped diffeomorphically onto $\mathbf{RP}^1$ by the projection π_n. In this case we fix any component $\mathcal{J}$ of A_0 and call it the *odd branch*.

For any parity of n we call the connected components of $\mathbf{R}A \setminus A_0$ *the ovals of* A. An oval V of A is called *odd* (resp. *even*) if a generic path in $\mathbf{R}\Sigma_n \setminus E_n$ relating V to $\mathcal{J}$ intersects A_0 in an odd (resp. even) number of points.

Definition. Let A be a nonsingular real algebraic curve on Σ_n satisfying the conditions (1)–(4). A *jump over* $\mathcal{J}$ is a pair of ovals (V_1, V_2) such that for any $p_1 \in V_1$, $p_2 \in V_2$, the ribbon $D = \pi_n^{-1}([q_1,q_2])$, $q_j = \pi_n(p_j)$, does not contain other ovals of A and the points p_1 and p_2 belong to different connected components of $D \setminus (E_n \cup \mathcal{J})$ (here we suppose that an orientation is fixed on $\mathbf{RP}^1$ and the segment $[q_1,q_2]$ is oriented from q_1 to q_2).

It is clear that the number of jumps is of the same parity as n.

A real algebraic curve A is called an *M-curve* if $\mathbf{R}A$ has $g+1$ connected components where g is the genus of A. It is called an $(M-r)$-curve if $\mathbf{R}A$ has $g+1-r$ connected components. If A is a curve of bidegree $(m,0)$ on Σ_n then $g = (m-1)(mn-2)/2$.

The main result of the paper is the following.

Theorem 1.1. *Let A be a real algebraic $(M-r)$-curve of bidegree $(m,0)$, $m = 2k+1$, on Σ_n satisfying the conditions (1)–(4). If n is divisible by 4, we suppose that $k = 1$. Let $\mathcal{J}$ be chosen as above and J be the number of jumps over $\mathcal{J}$. Let us denote the number of all ovals by λ, the number of odd ovals by λ_{odd}, and the number of even ovals by λ_{even}. We shall suppose that $\lambda > J > 0$.*

Then one has

$$|nk^2 - 3k + 1 + \varepsilon_{k,n} - r - J| \le r + 2\lambda_{\mathrm{odd}} + \begin{cases} k-1, & \text{if } n \text{ is odd} \\ 2(k-1), & \text{if } n \text{ is even} \end{cases} \tag{1}$$

$$|nk^2 - 3k + 1 + \varepsilon'_{k,n} - r - J| \le r + 2\lambda_{\mathrm{even}} + \begin{cases} k-1, & \text{if } n \text{ is odd} \\ 2(k-1), & \text{if } n \text{ is even} \end{cases} \tag{2}$$

where

$$\varepsilon_{n,k} = \frac{1+(-1)^k}{2}\cdot \operatorname{Re} i^{\,n-1} = \begin{cases} (-1)^{(n-1)/2} & \textit{if } k+1\equiv n\equiv 1 \mod 2, \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

$$\varepsilon'_{n,k} = \frac{1-3(-1)^k}{2}\cdot \operatorname{Re} i^{\,n-1} = \begin{cases} (-1)^{(n+1)/2} & \textit{if } k+1\equiv n\equiv 1 \mod 2, \\ 2\cdot(-1)^{(n-1)/2} & \textit{if } k\equiv n\equiv 1 \mod 2, \\ 0 & \textit{otherwise.} \end{cases} \tag{4}$$

This theorem will be proved in Section 9.

Now let us suppose that the curve A is *dividing*, or *of the type I*. This means that $A\setminus \mathbf{R}A$ consists of two halves which are mapped onto each other by the complex conjugation. An orientation of $\mathbf{R}A$ induced by the complex orientaion of one of the halves is called a *complex orientation*. There are two opposite complex orientation. Let us fix one of them. An oval V of A is called *positive* (resp. *negative*) if $[v]=-2[\mathcal{J}]$ (resp. $[v]=2[\mathcal{J}]$) in $H_1(\mathbf{R}\Sigma_n\setminus E_n)$. Let us denote the number of positive (resp. negative) ovals by λ_+ (resp. by λ_-). By Fiedler's alternating orientation rule [3]

$$J \ge |\lambda_+ - \lambda_-|. \tag{5}$$

Indeed, each sequence of ovals between successive jumps over $\mathcal{J}$ contrubutes at most 1 to the right hand side of (5).

Combination of (1), (5), and the formula for complex orientations from [7; Theorem 1.4A] allows to exclude some isotopy types of M-curves of degree 9 on $\mathbf{RP}^2$ (see Proposition 2.4).

2. Application for M-curves of degrees 7 and 9 on $\mathbf{RP}^2$

In this section we give details of the proofs of the results of [8].

2.1. Curves of degree 7.

Theorem 2.1. [8; Theorem 3]. *There does not exist an M-curve of degree 7 on $\mathbf{RP}^2$ with the complex scheme $\langle \mathcal{J}\sqcup 10_+\sqcup 3_-\sqcup 1_-\langle 1_-\rangle\rangle$.*

Proof. Blowing up a point inside the innermost oval, we are in the hypothesis of Theorem 1.1 with $n=1$, $k=3$, $r=0$, $\varepsilon_{n,k}=0$. Thus, (1) yeilds $|J-1|<2$ which contradicts to (5) because we have $\lambda_+=10$ and $\lambda_-=3$. $\square$

2.2. Curves of degree 9. Let A be an M-curve of degree 9 on $\mathbf{RP}^2$ whose isotopy type is $\langle \mathcal{J}\sqcup\alpha\sqcup 1\langle\beta\sqcup 1\langle\gamma\rangle\rangle\rangle$ with $\gamma\ge 1$ (see Figure 1). Abusing notation, we shall denote the set of empty ovals in the corresponding region by $\langle\alpha\rangle$, $\langle\beta\rangle$, or $\langle\gamma\rangle$. Let us denote the odd branch by $\mathcal{J}$ and the non-empty ovals by O_1 and O_2.

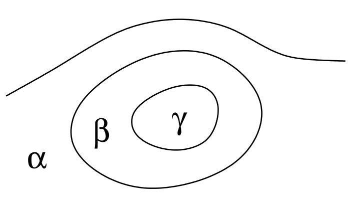


FIG. 1

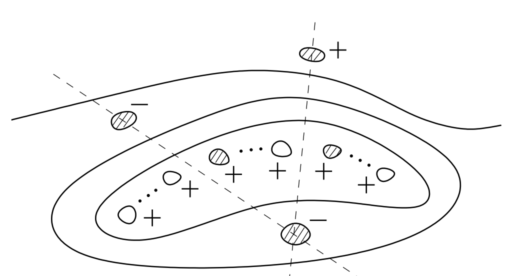


FIG. 2

For $v \in \langle\gamma\rangle$, let us denote by J_v the number of jumps through $\mathcal{J}$ in the pencil of lines centered at a point $p \in \operatorname{Int} v$ (i.e. the number of jumps for the transform of A on $\mathbf{RP}^2$ blown up at p). Then we have $n = 1$, $k = 4$, $r = 0$, $\varepsilon_{n,k} = 1$ and hence, (1) reads as

$$3 - 2\beta \le J_v \le 9 + 2\beta. \tag{6}$$

Lemma 2.2. *If ovals $v_1, v_2 \in \langle\gamma\rangle$ are separated by a line through ovals $v_3, v_4 \in \langle\alpha\rangle$ then $J_v = 1$ for any $v \in \langle\gamma\rangle$.*

Proof. Any conic through $v_1, \dots, v_4$ meets $O_1 \cup O_2$ at ≥ 8 points. Hence, if it meets one more empty oval, it cannot meet $\mathcal{J}$. Therefore, if we choose a point inside each empty oval then these points are vertices of a convex polygon which does not intersect $\mathcal{J}$ and the result follows. $\square$

Let the complex scheme of A be $\langle \mathcal{J} \sqcup \alpha_+ \sqcup \alpha_- \sqcup 1_{\varepsilon_2}\langle \beta_+ \sqcup \beta_- \sqcup 1_{\varepsilon_1}\langle\gamma\rangle\rangle\rangle$ where $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Let us set $\delta\alpha = \alpha^+ - \alpha^-$, $\delta\beta = \beta^+ - \beta^-$, $\delta\gamma = \gamma^+ - \gamma^-$, The Rohlin-Mishachev's formula for complex orientations implies

$$\delta\alpha + \varepsilon_2 + (1 - 2\varepsilon_2)(\delta\beta + \varepsilon_1) + (1 - 2\varepsilon_1 - 2\varepsilon_2)\delta\gamma = 8. \tag{7}$$

The formulas for complex orientations [7; Theorem 1.5A] imply

$$(\varepsilon_2 + 1)(\delta\beta + \delta\gamma) + (\varepsilon_1 + 1)\delta\gamma = (\varepsilon_2 + \varepsilon_1 + 2)^2/2. \tag{8}$$

The inequality (5) takes the form

$$J_v \ge |\delta\alpha + \delta\beta + \delta\gamma - \operatorname{sign} v|. \tag{9}$$

Combining it with (6), we obtain

$$|\delta\alpha + \delta\beta + \delta\gamma \mp 1| \le 9 + 2\beta \qquad \text{if } \gamma_\pm \ge 0. \tag{10}$$

Lemma 2.3. *If $|\delta\gamma| > 1$ and $\alpha > 0$ then $\beta > 0$.*

Proof. Suppose that $|\gamma_+ - \gamma_-| > 1$, $\alpha > 0$, and $\beta = 0$. Let us consider a pencil of lines through an oval from $\langle\alpha\rangle$. By Fiedler's alternating orientation rule, some two ovals from $\langle\gamma\rangle$ must be separated in this pencil by a line through another oval from $\langle\alpha\rangle$. By Lemma 2.2, this implies $J_v = 1$ which contradicts (6). $\square$

Theorem 2.4. *If an M-curve of degree nine is of the form $\langle \mathcal{J} \sqcup \alpha \sqcup 1\langle 1\langle\gamma\rangle\rangle\rangle$ where $\gamma > 0$, then its complex scheme is one of the following*

$$\langle \mathcal{J} \sqcup (\tfrac{\alpha+7}{2})_+ \sqcup (\tfrac{\alpha-7}{2})_- \sqcup 1_-\langle 1_-\langle (\tfrac{\gamma+1}{2})_+ \sqcup (\tfrac{\gamma-1}{2})_- \rangle\rangle\rangle,$$
$$\langle \mathcal{J} \sqcup (\tfrac{\alpha+7}{2})_+ \sqcup (\tfrac{\alpha-7}{2})_- \sqcup 1_\pm\langle 1_\mp\langle (\tfrac{\gamma-1}{2})_+ \sqcup (\tfrac{\gamma+1}{2})_- \rangle\rangle\rangle.$$

Proof. Only these complex schemes satisfy (7), (8), (10) and Lemma 2.3. $\square$

Corollary 2.5. [8; Theorems 1 and 2]. *If an M-curve of degree nine is of the form $\langle \mathcal{J} \sqcup \alpha \sqcup 1\langle 1\langle\gamma\rangle\rangle\rangle$, where $\gamma > 0$, then α is odd and $\alpha \ge 7$.* $\square$

Theorem 2.6. *The isotopy types* $\langle\mathcal{J}\sqcup 2\sqcup 1\langle 1\sqcup 1\langle 23\rangle\rangle\rangle$ *and* $\langle\mathcal{J}\sqcup 3\sqcup 1\langle 1\sqcup 1\langle 22\rangle\rangle\rangle$ *are not realizable by M-curves of degree nine.*

Proof. $\langle\mathcal{J}\sqcup 2\sqcup 1\langle 1\sqcup 1\langle 23\rangle\rangle\rangle$. It follows from (7) and (8), the complex scheme must be $\langle\mathcal{J}\sqcup 1_+\sqcup 1_-\sqcup 1_-\langle 1_-\sqcup 1_-\langle 13_+\sqcup 10_-\rangle\rangle\rangle$. Applying Fiedler's orientation alternating rule for a pencil of lines centered in the oval from $\langle\beta\rangle$, we see that the curve must be arranged with respect to some two lines as in Figure 2. Then a conic passing through the five shadowed ovals contradicts Bezout's theorem.

$\langle\mathcal{J}\sqcup 3\sqcup 1\langle 1\sqcup 1\langle 22\rangle\rangle\rangle$. It follows from (7) and (8), that the complex scheme must be $\langle\mathcal{J}\sqcup 1_+\sqcup 2_-\sqcup 1_-\langle 1_+\sqcup 1_-\langle 12_+\sqcup 10_-\rangle\rangle\rangle$. Let us consider a pencil of lines centered in an oval from $\langle\alpha\rangle$. By Fiedler's orientation alternating rule, the ovals $\langle\gamma\rangle$ must be separated by a line through an oval from $\langle\beta\rangle\cup\langle\alpha\rangle$. It cannot be the oval from $\langle\beta\rangle$ because the latter is positive. Hence, the hypothesis of Lemma 2.3 are satisfied, and we have $J_v = 1$ for any $v\in\langle\gamma\rangle$. This contradicts to (9). □

Remark 2.7. In [7; Corollary 1.8], we announced the nonrealizability of 9 isotopy types by M-curves of 9-th degree. The proof given in [7] was based on a wrong assertion of [7; Lemma 1.9] (see [7; *Erratum*]). However, Corollary 2.5 and Theorem 2.6 imply that the statement of [7; Corollary 1.8] was correct.

Remark 2.8. We give Theorem 2.6 not as an application of Theorem 1.1 (which is not used in the proof) but just by the reasons explained in Remark 2.7.

3. Signature, determinant and skein-relation

Recall some definitions. Let B be a real symmetric matrix and $D = QBQ^T$ its diagonalization. *The signature* $\operatorname{Sign} B$ and *nullity* $\operatorname{Null} B$ of B are defined as the sum of signs of the diagonal entries and the number of zeros on the diagonal of D.

A Seifert surface of an oriented link L in the 3-sphere S^3 is a conncted oriented surface $X\subset S^3$ whose boundary (taking into account the orientation) is L. Let $x_1,\dots,x_n$ be a base in $H_1(X,\mathbf{Z})$. *A Seifert matrix* is the matrix of the linking numbers $\operatorname{lk}(x_i,x_j^+)$ where x_j^+ is the result of a small shift of the cycle x_j along a positive normal field to X. The *signature* $\operatorname{Sign} L$ and the *nullity* $\operatorname{Null} L$ of L are defined by[1] $\operatorname{Sign} L = \operatorname{Sign}(V+V^T)$, $\operatorname{Null} L = \operatorname{Null}(V+V^T)$ where V is a Seifert matrix of L.

The *Conway potential function* of a link L is defined as $\Omega_L(t) = \det(t^{-1}V - tV^T)$ where V is a Seifert matrix of L. By definition, $\Omega_L(t) = 1$ if L is the trivial knot. We define the *determinant* of L as $\det L = \Omega_L(i)$ where $i=\sqrt{-1}$. It is known that $\Omega_L(t)$ is a link invariant and the following *skein-relation* holds (see [4]). Let L_-, L_0, and L_+ be links whose diagrams coincide everywhere except of some disk where they look as in Fig. 4. Then

$$\Omega_{L_+}(t) - \Omega_{L_-}(t) = (t - t^{-1})\Omega_{L_0}(t). \tag{11}$$

Substituting $t = i$ into (11), we obtain

$$\det L_+ - \det L_- = 2i\det L_0. \tag{12}$$

[1] The nullity of L is defined usually as $1 + \operatorname{Null}(V+V^T)$.

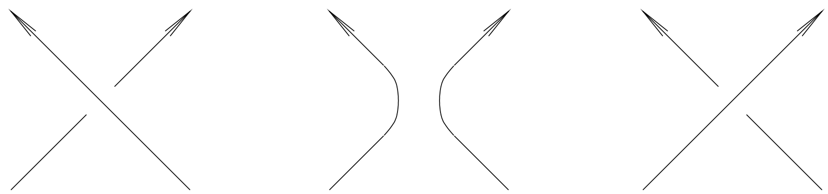


FIG. 4

Let L and L' be oriented links. Say that L' is obtained from L by a *band-attachement* if $L' = (L \setminus b(\partial I \times I)) \cup b(I \times \partial I)$ where $I = [0,1]$ and $b : I^2 \to S^3$ is an embedding such that $L \cup b(I^2) = b(\partial I \times I)$ and the orientations of L and L' coincide on $L \cap L'$. For instance, L_+ and L_- in Fig. 4 are obtained from L_0 by a band-attachement. It is clear that if L' is obtained from L by a band-attachement then L also can be obtained from L' by a band-attachement.

Lemma 3.1. (Conway). *Let a link L' be obtained by a band-attachement from a link L.*

(a). If $\det L \neq 0$ *then*

$$\operatorname{Sign} L' = \operatorname{Sign} L + \operatorname{sign} \frac{i \det L'}{\det L} \tag{13}$$

(b). $|\operatorname{Null} L' - \operatorname{Null} L| + |\operatorname{Sign} L' - \operatorname{Sign} L| = 1.$

Remark. $\det L \in \mathbf{Z}$ (resp. $\in i\mathbf{Z}$) if the number of components of L is odd (resp. even) and the band attachement changes the parity of the number of components. Hence, the fraction in (13) is real.

Proof. Let $b : I \times I \to S^3$ be the attached band. One can choose a Seifert surface X for L such that $X \cap b(I^2) = b(\partial I \times I)$. Then $X' = X \cup b(I \times I)$ is a Seifert surface for L' and one can choose a base of $H_1(X')$ by adding one element to a base of $H_1(X)$. Let V and V' be the corresponding Seifert matrices. Then V is a principal $n \times n$-minor of an $(n+1) \times (n+1)$-matrix V'. This implies the part (b). To prove the part (a), note that

$$\operatorname{Sign} L' - \operatorname{Sign} L = \operatorname{sign} \frac{\det(V' + V'^T)}{\det(V + V^T)} = \operatorname{sign} \frac{i^{n+1} \det L'}{i^n \det L}$$

$\square$

Corollary 3.2. *Let each of two links L' and L'' be obtained by a band-attachement from the same link L. If* $\det L' = \det L'' \neq 0$ *then* $\sigma(L'') = \sigma(L')$. $\square$

4. ITERATED TORUS LINKS AND THEIR SPLICE DIAGRAMS

Let L be an oriented link in S^3 and S a component of L. Let T be a tubular neighbourhood of S which does not intersect $L \setminus S$. Let p and q be coprime integers (in particular, if one of them is 0 then the other is ± 1). Let d be a positive integer. We say that L' is obtined from L by a (dp, dq)*-cabling along S with the core removed* (resp. *remained*) if $L' = (L \setminus S) \cup S'$ (resp. $L' = L \cup S'$) where S' is a disjoint union of circles $S' = S_1 \cup \dots \cup S_d \subset \partial T$ such that for each j we have $[S_j] = p[S]$ in

$H_1(T)$ and $\mathrm{lk}(S_j, S') = q$. A link is called an *iterated torus link* or a *solvable link* if it can be successively obtained from an unknot by these operations and maybe reversing the orientations of some components.

To work with iterated toric links, we shall use the language of splice diagrams introduced by Eisenbud and Neumann in [2]. A splice diagram is a tree Γ which has no vertices of valence 2 and which is decorated as follows. Some of leaves (i.e. vertices of valence 1) of Γ are depicted as arrowheads, a sign ± 1 is attrubuted to each arrowhead, and an integer $w(v,e)$ called the *edge weight* is attributed to each pair (v,e) where v is a *node* (i.e. a vertex of valence ≥ 3) and e is an edge incident to v. Depicting splice diagrams, we write $w(e,v)$ at the beginning of e at v; if the sign of an arrowhead is $+1$ then we do not write it. The arrowhead vertices of the splice diagram of a link correspond to the link components.

We shall define the splice diagram of an iterated torus link inductively as follows. The splice diagram of the unknot is $\bullet\!\!\longrightarrow$. Let $\boxed{\Gamma}\xrightarrow{\varepsilon}$ with $\varepsilon = \pm 1$ be the splice diagram of L and the depicted arrowhead corresponds to S. If L' is obtained from L by a (dp, dq)-cabling along S with the core removed (resp. remained) then the splice diagram of L' is as in Figure 10 (resp. in Figure 11). Reversing of the orientation of a link component corresponds to changing of the sign of the corresponding arrowhead.

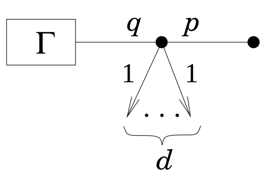


FIG. 10

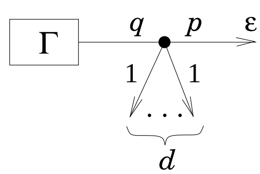


FIG. 11

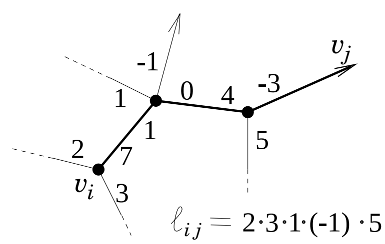


FIG. 12

A splice diagram is not determined by a link but Eisenbud and Neumann [2] described an equivalence relation between splice diagrams such that the equivalence calss of a splice diagram is uniquely determined by a link. Thay gave also a formula for the Alexander polynomial of an iterated torus link in terms of its splice diagram (in fact, this was done in [2] for graph links in homology spheres). Neumann [6] proved an analogous formula (only in the case of fiberable links S^3) for the Conway polynomial. To write this formula, we need the following notation.

Let Γ be a splice diagram. Let us denote its vertices by $v_1, \dots, v_n, v_{n+1}, \dots, v_k$ with $v_1, \dots, v_n$ being arrowheads (denote their signs by $\varepsilon_1, \dots, \varepsilon_n$) and $v_{n+1}, \dots, v_k$ being the remaining vertices (denote their valences by $\delta_{n+1}, \dots, \delta_k$). For any two distinct vertices v_i and v_j of Γ let us denote by s_{ij} the simple path in Γ joining v_i to v_j, including v_i and v_j, and define (see Figure 12)

$$\ell_{ij} = \prod_{v \in s_{ij}, e \not\subset s_{ij}} w(v,e), \quad i,j = 1, \dots, k; \qquad m_i = \sum_{j=1}^{n} \ell_{ij}\varepsilon_j, \quad i = n+1, \dots, k.$$

Theorem 4.2. *Let L be a solvable link and Γ its splice diagram.*

a). [2]. L is fiberable if $m_i \neq 0$ for any $i = n+1, \dots, k$.

b). [6]. *If L is fiberable then*

$$\Omega_L(t) = \varepsilon_1 \dots \varepsilon_n \cdot (t - t^{-1}) \prod_{i=n+1}^{k} \left(t^{m_i} - t^{-m_i}\right)^{\delta_i - 2} \tag{14}$$

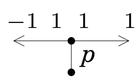
Corollary 4.3. *Let L be a solvable link and Γ its splice diagram. Suppose that $m_i \neq 0$ as soon as $\delta_i = 1$ (i.e. no denominator in the right hand side of* (14) *vanishes). Then the equality* (14) *holds.*

Proof. If $m_i \neq 0$ for all $i = n+1, \dots, k$ then the result follows from Theorem 4.2. If $m_i = 0$ for some i then $\Omega_L = 0$ by Eisenbud-Neumann formula for the multivariable Alexander polynomial [2; Theorem 12.1]. $\square$

Remark 4.4. Recently, Cimasoni [1] proved the analogue of Eisenbud-Neumann formula for the multivariable Conway potential function for any graph link L in a homology sphere:

$$\nabla_L(t_1, \dots, t_n) = \varepsilon_1 \dots \varepsilon_n \prod_{i=n+1}^{k} \left(t_1^{\ell_{i1}\varepsilon_1} \dots t_n^{\ell_{in}\varepsilon_n} - t_1^{-\ell_{i1}\varepsilon_1} \dots t_n^{-\ell_{in}\varepsilon_n}\right)^{\delta_i - 2}$$

(the terms $t_1^0 \dots t_n^0 - t_1^0 \dots t_n^0$ should be formally cancelled against each other before being set equal to zero). Since $\Omega_L(t) = (t - t^{-1})\nabla_L(t, \dots, t)$, this formula allows one to compute Ω_L in the cases not covered by Corollary 4.3, i.e. in the cases when (14) does not make sense.

For example (cp. [2; p. 97]), if $\Gamma =$ 

then $\nabla_L = -(u^p - u^{-p})/(u - u^{-1}) = -(u^{p-1} + \dots + u + 1)$ where $u = t_1/t_2$. Hence, $\nabla_L(t,t) = -p$ and $\Omega_L = (t^{-1} - t)p$ which well agrees with the fact that a Seifert matrix of L is the 1×1-matrix (p).

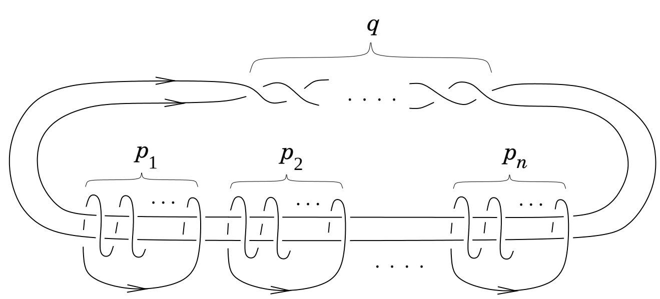


FIG. 13

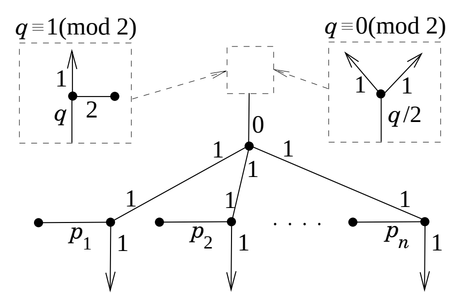


FIG. 14

The following lemma provides an example of application of the above formulas. It will be used in the next section.

Lemma 4.5. *Let L be the link in Fig. 13 where $q, p_1, \dots, p_n \in \mathbf{Z}$ (if some of the parameters $q, p_1, \dots, p_n$ are negative, the corresponding positive crossings in Fig. 13 should be replaced with negative ones; if $p_j = 0$ then the corresponding component is a simple closed curve disjoint from the rest of the picture). If $n > 1$ then* $\det_L = 0$.

Proof. If $p_j = 0$ for some j then the link splits (has a component separated by a 2-sphere from the others) and $\Omega_L = 0$. Assume that $p_j \neq 0$ for all j. Then the splice diagram of L is as in Fig. 14. By Theorem 4.2, L is fiberable unless

$$q + p_1 + \dots + p_n = 0 \tag{15}$$

and if it is fiberable then $\Omega_L(t) = (t - t^{-1})\prod_v \omega_v(t)$ where v runs over the non-arrowhead vertices of valence $\neq 2$. Let u be the vertex of valence $n+1$ (whose outcoming edges are weighted by $0, 1, \dots, 1$). If $n > 1$ then $\omega_u(t) = (t^2 - t^{-2})^{n-1}$, hence, $\omega_u(i) = 0$. Thus, if $n > 1$ and (15) does not hold then $\det_L = 0$.

In the case when (15) holds, set $L_0 = L$ and let $L_\pm$ be the link obtained from L by replacing q with $q \pm 1$. Then (15) does not hold for $L_\pm$, and we have $\det L_\pm = 0$. Hence, $\det L = 0$ by (12). $\square$

5. Link theoretical lemma

Let S be an oriented unknotted circle in S^3 and L an oriented link which contains S as a component. Suppose that L' is a link whose diagram is obtained from a diagram of L by simultaneous replacing of a negative crossing with a positive one and a positive crossing with a negative one where each of the two crossings involves a segment of S and a segment of $L \setminus S$ (see Fig. 15 where S is thicker than $L \setminus S$).

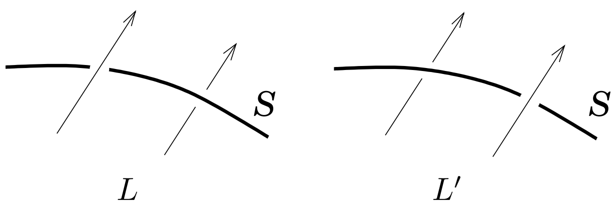


Fig. 15

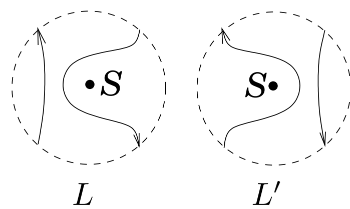


Fig. 16

In other words, L' is obtained from L as follows. Choose disjont embedded disks D_1 and D_2 such that

(1) $a_i := D_i \cap (L \setminus S)$ is a segment of ∂D_i, $i = 1, 2$;
(2) each D_i meets S transversally at a single point;
(3) the signs of the intersections are opposite (i.e. $D_1 \cdot S = -D_2 \cdot S$) where the orientation of D_i is chosen so that the orientation of a_i induced from L coincides with that induced from ∂D_i.

Then $L' = (L \setminus (a_1 \cup a_2)) \cap (a_1' \cup a_2')$ where $a_i' = \overline{\partial D_i \setminus a_i}$.

Lemma 5.1. *Let q be any integer and $L_{2,q}$ (resp. $L'_{2,q}$) be the $(2, q)$-cabling of L (resp. of L') along S with the core removed. Then:*
a). $\det L'_{2,q} = \det L_{2,q}$*;*
b). If $\det L_{2,q} \neq 0$ *then* $\operatorname{Sign} L'_{2,q} = \operatorname{Sign} L_{2,q}$.

Proof. a). Let us identify S^3 with $\mathbf{R}^3 \cup \{\infty\}$ so that S is a line and the projection of $L \setminus S$ onto a plane along the line S is non-degenerate. One can assume that there exists a disk $D \subset \mathbf{R}^2$ centered at the projection of S such that the diagrams of $L \setminus S$ and $L' \setminus S$ (with respect to the chosen projection) coincide outside D and have the form shown in Fig. 16 inside D.

Let us use the induction by the number of crossings of the diagram of $L \setminus S$. Suppose that there are no crossings. If $L \setminus S$ is connected then $L' = L$. Otherwise, each of $L_{2,q}$ and $L'_{2,q}$ has the form shown in Fig. 13 with $p_j = 0, \pm 1$ and $n > 1$, hence, $\det L_{2,q} = \det L'_{2,q} = 0$ by Lemma 4.5.

Now suppose that the number of crossings of the diagram of $L \setminus S$ is N and suppose that the statement of the lemma is proved for all diagrams with less than

N crossings. Denote by π the projection $\mathbf{R}^3 \to \mathbf{R}^2$ (recall that $\pi(S)$ is a single point). Let $L_1, \dots, L_n$ be the copmonents of $L \setminus S$. On each L_j, let us choose a point x_j such that $\pi(x_j)$ lies on the boundary of the unbounded component of $\mathbf{R}^2 \setminus \pi(L_j)$ and $\pi(x_j)$ is not a double point of $\pi(L \setminus S)$. Let $f : (L \setminus S) \setminus \{x_1, \dots, x_n\} \to \mathbf{R}$ maps homeomorphically each $L_j \setminus x_j$ onto the interval $(j-1, j)$. Let $x, y \in L$ be such that $\pi(x) = \pi(y)$ and $f(x) < f(y)$. Say that the corresponding crossing of the diagram of $L \setminus S$ is *monotone* if y is higher than x.

Consider the family of all diagrams which differ from the diagram of L by changing the signs of crossings (the number of such diagrams is 2^N). Let us prove the statement of the lemma for the diagrams from this family using the induction by the number of non-monotone crossings (with respect to the same choice of $x_1, \dots, x_n$).

Base case of the induction. Suppose that all the crossings of L are monotone. Then each of $L_{2,q}$ and $L'_{2,q}$ has the form shown in Fig. 13 where p_j is the linking number $\operatorname{lk}(L_j, S)$ (if $p_j = 0$ then L_j should be shown in Fig. 13 by a simple curve disjoint from the rest of the picture). Hence, $L = L'$ for $n = 1$ and, by Lemma 4.5, we have $\det L_{2,q} = \det L'_{2,q} = 0$ for $n > 1$.

Step of the induction. Consider a non-monotone crossing of the diagram of $L \setminus S$. Let it be positive (the case of a negative non-monotone crossing can be treated similarly). Set $L_+ = L$, $L'_+ = L'$ and let L_0 and L_- (resp. L'_0 and L'_-) be obtained from L_+ (resp. L'_+) by changing the crossing according to Fig. 4. By the induction hypothesis, we have $\det(L_0)_{2,q} = \det(L'_0)_{2,q}$ (because the number of crossings of L_0 is less than that of L_+) and $\det(L_-)_{2,q} = \det(L'_-)_{2,q}$ (because the number of non-monotone crossings of L_- is less than that of L_+). Hence, $\det(L_+)_{2,q} = \det(L'_+)_{2,q}$ by (12).

b). Follows from (a) and Corollary 3.2. $\square$

6. Computation of some braid determinants

Notation 6.1. Let us denote the standard generators of the braid group B_m by $\sigma_1, \sigma_2, \dots, \sigma_{m-1}$ and let us set:

$$\pi_{k,l} = \begin{cases} \sigma_k \sigma_{k+1} \dots \sigma_l, & k < l, \\ \sigma_k \sigma_{k-1} \dots \sigma_l, & k > l, \\ \sigma_k, & k = l, \end{cases} \qquad \tau_{k,l} = \begin{cases} \pi^{-1}_{l,k+1} \pi_{k,l-1}, & k < l, \\ \pi^{-1}_{l,k-1} \pi_{k,l+1}, & k > l, \\ 1, & k = l, \end{cases}$$

$\Delta_k = \pi_{1,k-1} \pi_{1,k-2} \dots \pi_{1,2} \sigma_1$.

For positive interegs n, J, k such that $n \equiv J \mod 2$ and non-negative integers $\alpha_1, \dots, \alpha_J$ let us set $m = 2k+1$ and define the braid $b = b^J_{n,k}(\alpha_1, \dots, \alpha_J) \in B_m$ as

$$b = b_1 \dots b_J \, \Delta^n, \qquad b_j = \begin{cases} \sigma_{k-1}^{-\alpha_j} \tau_{k-1,k}, & \text{if } j \text{ is odd}, \\ \sigma_k^{-\alpha_j} \tau_{k,k-1}, & \text{if } j \text{ is even}. \end{cases}$$

If, moreover, $k \geq 2$, we define the braid $c = c^J_{n,k}(\alpha_1, \dots, \alpha_J) \in B_{2k+1}$ as

$$c = c_1 \dots c_J \, \Delta^n, \qquad c_j = \begin{cases} \sigma_{k-2}^{-\alpha_j} \tau_{k-2,k+1}, & \text{if } j \text{ is odd}, \\ \sigma_{k+1}^{-\alpha_j} \tau_{k+1,k-2}, & \text{if } j \text{ is even}. \end{cases}$$

For $\vec{\alpha} = (\alpha_1, \dots, \alpha_J)$, $\alpha = \alpha_1 + \dots + \alpha_J$, let us set

$$\tilde{b}^J_{n,k}(\vec{\alpha}) = i^\alpha \det b^J_{n,k}(\vec{\alpha}), \qquad \tilde{c}^J_{n,k}(\vec{\alpha}) = i^\alpha \det c^J_{n,k}(\vec{\alpha}), \tag{16}$$

Lemma 6.2. *a). Let $n, k, J \ge 1$ and $n \equiv J \mod 2$. Then*

$$\det b^J_{n,k}(1,\dots,1) = \begin{cases} 4 & \text{if } n \equiv J+2 \equiv 0 \mod 4 \text{ and } k=1,\\ 4^k & \text{if } n+2 \equiv J \equiv 0 \mod 4,\\ (-2i^n)^{k+1} & \text{if } n \text{ is odd and } J \equiv n+2k \mod 4,\\ 0 & \text{otherwise.} \end{cases}$$

b). Let $n, J \ge 1$, $k \ge 2$, and $n \equiv J \mod 2$. Then

$$\det c^J_{n,k}(1,\dots,1) = \begin{cases} 4^k & \text{if } n+2 \equiv J \equiv 0 \mod 4\\ -(-2i^n)^{k+1} & \text{if } n \text{ is odd and } J \equiv n+2k+2 \mod 4,\\ 0 & \text{otherwise.} \end{cases}$$

Proof. a). Let us show that the closure L of $b^J_{n,k}$ is an iterated torus link and its splice diagram is shown in Fig. 18. Indeed, let us apply the procedure form Section 4. We start with the k-th string and we add the strings number $1,\dots,k-1$, $k+3,\dots,m$ as the $(2,n)$- or $(1,n/2)$- (according to the parity of n) -cables along it (see Fig. 17). Then we add the $(k+1)$-th string as the $(1,(n-J)/2)$-cable along the k-th string and finally, we add the $(k+2)$-th string as the $(1,(n+J)/2)$-cable along the $(k+1)$-th string.

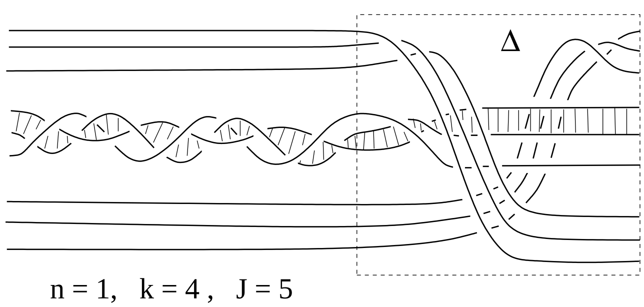


FIG. 17. THE BRAID $b^J_{n,k}$

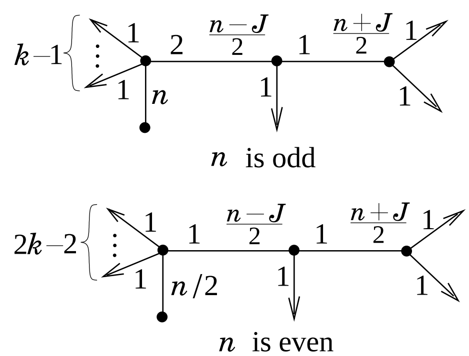


FIG. 18. THE S.-D. OF $b^J_{n,k}$

By Corollary 4.3, we have

$$\Omega_L(t) = \frac{(t-t^{-1})\cdot\omega_1^{k-1}\cdot(t^{m_2}-t^{-m_2})(t^{m_3}-t^{-m_3})}{t^m - t^{-m}} \tag{17}$$

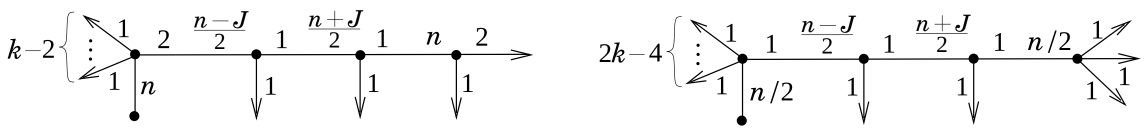
where

$$\omega_1 = (t^{nm} - t^{-nm}) \text{ if } n \text{ is odd}, \quad \omega_1 = (t^{nm/2} - t^{-nm/2})^2 \text{ if } n \text{ is even.}$$

and

$$m_2 = n(k-1) + \tfrac{3}{2}(n-J) = \tfrac{mn}{2} - \tfrac{3}{2}J,$$
$$m_3 = n(k-1) + \tfrac{1}{2}(n-J) + (n+J) = \tfrac{mn}{2} + \tfrac{1}{2}J.$$

b). The proof is similar to that of the part (a). The closure L of $c^J_{n,k}$ is an iterated torus link and its splice diagram is shown in Fig. 19. Indeed, if we remove



n is odd $\qquad$ n is even

FIG. 19. THE SPLICE DIAGRAM OF $c^J_{n,k}$

the $(k+2)$-th and $(k+3)$-th strings, we obtain the braid $b^2_{n,k-2}$ whose splice diagram is computed above and these two strings can be considered as the $(2,n)$- or $(1,n/2)$-cables along the $(k+1)$-th string.

Thus, by Corollary 4.3, we obtain

$$\Omega_L(t) = \frac{(t-t^{-1})\cdot\omega_1^{k-2}\cdot(t^{m_2}-t^{-m_2})(t^{m_3}-t^{-m_3})\cdot\omega_4}{t^m-t^{-m}}$$

where

$$\begin{aligned} &\omega_1 = (t^{nm}-t^{-nm}), &&\omega_4 = t^{2m_4}-t^{-2m_4} &&\text{if } n \text{ is odd,}\\ &\omega_1 = (t^{nm/2}-t^{-nm/2})^2, &&\omega_4 = (t^{m_4}-t^{-m_4})^2 &&\text{if } n \text{ is even.} \end{aligned}$$

and

$$\begin{aligned} m_2 &= n(k-2)+\tfrac52(n-J) = \tfrac{mn}{2}-\tfrac52 J,\\ m_3 &= n(k-2)+\tfrac12(n-J)+2(n+J) = \tfrac{mn}{2}+\tfrac32 J,\\ m_4 &= n(k-2)+\tfrac12(n-J)+\tfrac12(n+J)+\tfrac32 n = \tfrac{mn}{2} \end{aligned}$$

(we see a posteriori that $\omega_4=\omega_1$). $\square$

Lemma 6.3. *a). If n is odd and $k\ge 1$ then* $\det b^1_{n,k}(0) = (2i^n)^k$.

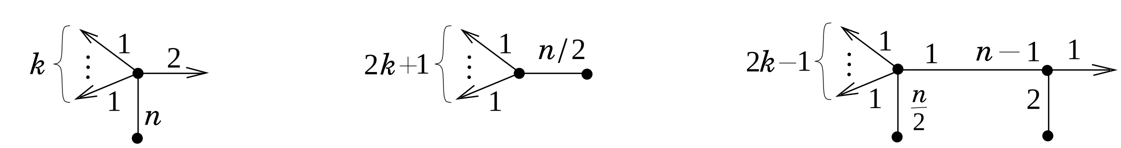
b). If n is even and $k>0$ then

$$\det b^2_{n,k}(0,0) = i\,\det b^2_{n,k}(1,0) = (2-2i^n)^k = \begin{cases} 4^k, & n\equiv 2 \mod 4,\\ 0, & n\equiv 0 \mod 4,\end{cases}$$

c). If $k\ge 2$ then the braids $c^1_{n,k}(0)$, $c^2_{n,k}(0,0)$, and $c^2_{n,k}(1,0)$ are conjugate to $b^1_{n,k}(0)$, $b^2_{n,k}(0,0)$, and $b^2_{n,k}(1,0)$ respectively.

Proof. a). Since $b^1_{n,k}(0) = \sigma_k^{-1}\sigma_{k-1}\Delta^n = \sigma_k^{-1}\Delta^n\sigma_k$ is conjugate to Δ^n, it represents the toric link whose diagram is depicted in Figure 20. Hence,

$$\Omega_{b^1_{n,k}(0)}(t) = (t-t^{-1})\cdot\frac{(t^{mn}-t^{-mn})^k}{t^m-t^{-m}}, \qquad \text{where } m = 2k+1.$$



$b^1_{n,k}(0) = \sigma_k^{-1}\Delta^n\sigma_k$ $\qquad$ $b^2_{n,k}(0,0) = \Delta^n$ $\qquad$ $b^2_{n,k}(1,0)$

FIG. 20

b). The links represented by the braids $b^2_{n,k}(0,0)=\Delta^n$ and $b^2_{n,k}(1,0)$ are solvable. Their splice diagrams are as in Figure 20, so we have

$$\Omega_{b^2_{n,k}(0,0)}(t)=(t-t^{-1})\cdot\frac{(t^{mn/2}-t^{-mn/2})^{2k}}{t^m-t^{-m}},$$

$$\Omega_{b^2_{n,k}(1,0)}(t)=(t-t^{-1})\cdot\frac{(t^{mn/2}-t^{-mn/2})^{2k-1}}{t^m-t^{-m}}\cdot\frac{t^{2m_2}-t^{-2m_2}}{t^{m_2}-t^{-m_2}}$$

where $m_2=(2k-1)\frac{n}{2}+(n-1)=\frac{mn}{2}-1>0$.

c). Easy to check. $\square$

7. Skein systems of cyclically symmetric polynomials

Definition 7.1. Let $\nu=1$ or 2. A sequence of polynomials $\{f_J(x_1,\dots,x_J)\}$, $J=\nu,\nu+2,\nu+4,\dots$, is called a *skein system of cyclically symmetric polynomials of the parity* ν (or, just a *skein system of parity* ν) if the following conditions hold

(i) $\deg_{x_1} f_J\le 1$;
(ii) $f_J(x_1,\dots,x_J)=f_J(x_J,x_1,\dots,x_{J-1})$;
(iii) $f_J(x_1,0,x_3,\dots,x_J)=f_{J-2}(x_1+x_3,x_4,\dots,x_J)$ for $J\ge\nu+2$.

Lemma 7.2. *Let $f_J=\tilde b^J_{n,k}$ or $\tilde c^J_{n,k}$ (see Section 6).*

a). If we fix positive integers n,J,k such that $n\equiv J \mod 2$ then $f_J(\alpha_1,\dots,\alpha_J)$ are values at integral points of a polynomial (we denote it also f_J).

b). For fixed $n,k>0$, the sequences $\{f_J\}_{J>0,\,J\equiv n(2)}$ is a skein system of the parity n.

Proof. When all x_j are integer, (ii) and (iii) are evident and it follows from (12) that f_J is linear with respect to each variable. $\square$

Lemma 7.3. *Let $\mathcal F=\{f_J\}$ be a skein system of parity ν. Let $c_J=f_J(1,\dots,1)$.*

If $\nu=1$ then $\mathcal F$ is uniquely determined by the sequence $\{c_1,c_3,c_5,\dots\}$ and the number $c_0=f_1(0)$.

If $\nu=2$ then $\mathcal F$ is uniquely determined by the sequence $\{c_2,c_4,c_6\dots\}$ and the numbers $c_0=f_2(0,0)$, $c_1=f_2(1,0)$.

Proof. Suppose we proved the uniqueness of f_{J-2}. By (i) – (ii), f_J is determined by its values at the vertices of the unit cube. The value at $(1,\dots,1)$ is c_J. The values at other vertices can be expressed by (ii) and (iii) in terms of f_{J-2}. $\square$

Let us define symmetric $(J\times J)$-matrices $A^\pm_J(x_1,\dots,x_J)$ as follows. Let $E_{i,j}$ be the $(J\times J)$-matrix whose (k,l)-th entry is $\delta_{ki}\delta_{jl}$. Set

$$A^\pm_J(x_1,\dots,x_J)=-2\left(\sum_{i=1}^J x_iE_{i,i}\right)+\left(\sum_{i=1}^{J-1}E_{i,i+1}+E_{i+1,i}\right)\pm(E_{1,J}+E_{J,1}).$$

Thus,

$$A^\pm_1=(\pm2-2x_1),\ A^+_2=\begin{pmatrix}-2x_1 & 2\\ 2 & -2x_2\end{pmatrix},\ A^-_2=\begin{pmatrix}-2x_1 & 0\\ 0 & -2x_2\end{pmatrix},$$

$$A^\pm_3=\begin{pmatrix}-2x_1 & 1 & \pm1\\ 1 & -2x_2 & 1\\ \pm1 & 1 & -2x_3\end{pmatrix},\ A^\pm_4=\begin{pmatrix}-2x_1 & 1 & 0 & \pm1\\ 1 & -2x_2 & 1 & 0\\ 0 & 1 & -2x_3 & 1\\ \pm1 & 0 & 1 & -2x_4\end{pmatrix},\ \dots$$

Lemma 7.4. $\det A_J^-(x_1,\dots,x_J) = \det A_J^+(x_1,\dots,x_J) + (-1)^J\cdot 4$.

Proof. The determinant of the matrix obtained from A_J^+ by replacing the $(1,J)$-th and $(J,1)$-th entry with u, is a quadratic function of u whose linear term is equal to $(-1)^{J+1}\cdot 2u$. $\square$

Lemma 7.5. $\det A_J^\pm(x_1,0,x_3,\dots,x_J) = -\det A_{J-2}^\mp(x_1+x_3,x_4,\dots,x_J)$.

Proof. Indeed, we have $\det A_J^\pm(x_1,0,x_3,\dots,x_J)\ =$

$$= \begin{vmatrix} -2x_1 & 1 & 0 & \dots & \pm1 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & -2x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ \pm1 & 0 & 0 & & \end{vmatrix} = \begin{vmatrix} 0 & 1 & x_1 & \dots & \pm1 \\ 1 & 0 & 1 & \dots & 0 \\ x_1 & 1 & -2x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ \pm1 & 0 & 0 & & \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & 1 & x_1 & \dots & \pm1 \\ 1 & 0 & 0 & \dots & 0 \\ x_1 & 0 & -2(x_3+x_1) & \dots & \mp1 \\ \vdots & \vdots & \vdots & & \\ \pm1 & 0 & \mp1 & & \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\cdot\det A_{J-2}^\mp(x_1+x_3,x_4,\dots,x_J). \quad \square$$

Let us denote $a_J^\pm = \begin{cases} \pm\det A_J^\pm & \text{if } J\equiv 0 \text{ or } 1 \mod 4, \\ \mp\det A_J^\mp & \text{if } J\equiv 2 \text{ or } 3 \mod 4. \end{cases}$

Corollary 7.6.
$\{a_1^+,a_3^+,a_5^+\dots\}$ and $\{a_1^-,a_3^-,a_5^-\dots\}$ are odd skein systems;
$\{a_2^+,a_4^+,a_6^+\dots\}$ and $\{a_2^-,a_4^-,a_6^-\dots\}$ are even skein systems; $\square$

Lemma 7.7. *a).* $\det A_J^+(1,\dots,1)=0$;

b). $\det A_J^-(1,\dots,1) = (-1)^J\cdot 4$.

c). If $x_j\ge 1,\dots,x_J\ge 1$ and $(x_1,\dots,x_J)\ne(1,\dots,1)$ then

$$\operatorname{sign}\det A_J^+(x_1,\dots,x_J) = (-1)^J.$$

d). If $x_j\ge 1,\dots,x_J\ge 1$ then $\operatorname{sign}\det A_J^-(x_1,\dots,x_J) = (-1)^J$.

Proof. a). $\det A_J^+=0$ because the sum of the rows is zero;

b). Follows from (a) and Lemma 7.4.

c). Set $A_J^+(x_1,\dots,x_J) = A+D$ where $A = A_J^+(1,\dots,1)$ and $D=\operatorname{diag}(2-2x_1,\dots,2-2x_J)$. Let $e_1,\dots,e_J$ be the standard base of $\mathbf{R}^J$. Then A defines a quadratic form on $\mathbf{R}^J$ whose restriction onto $\langle e_1,\dots,e_{J-1}\rangle$ is negative definite (this is the Cartan matrix of the type A) and the kernel of the form A is generated by the vector $v=e_1+\dots+e_J$. The diagonal form D is non-positive and it is negative on v. Thus, the form $A+D$ is negative definite.

d). Follows from (b) and the fact that the principal $(J-1)\times(J-1)$-minors are negative definite (see the proof of (c)). $\square$

Corollary 7.8. *The skein systems from Corollary 7.6 satisfy the following initial conditions.*

$$\begin{array}{lll}
a_1^+(0)=2, & a_J^+(1,\dots,1)=2(i^{J+1}+1), & J=1,3,5,\dots\\
a_1^-(0)=2, & a_J^-(1,\dots,1)=2(i^{J-1}+1), & J=1,3,5,\dots\\
a_2^+(0,0)=a_2^+(1,0)=0, & a_J^+(1,\dots,1)=2(i^{J}-1), & J=2,4,6,\dots\\
a_2^-(0,0)=a_2^-(1,0)=-4, & a_J^-(1,\dots,1)=-2(i^{J}+1), & J=2,4,6,\dots
\end{array}\qquad\square$$

Corollary 7.9. *Let n, k, J be positive integers and let $\vec\alpha=(\alpha_1,\dots,\alpha_J)$, $\alpha_j\ge 0$. If $n\equiv 0 \mod 4$ and $k>1$ then*

$$\tilde b^J_{n,k}(\vec\alpha)=\tilde c^J_{n,k}(\vec\alpha)=0.$$

If $n\equiv 0 \mod 4$ and $k=1$ then

$$\tilde b^J_{n,k}(\vec\alpha)=a_J^+(\vec\alpha).$$

If $n\equiv 2 \mod 4$ then

$$\tilde b^J_{n,k}(\vec\alpha)=\tilde c^J_{n,k}(\vec\alpha)=-4^{k-1}a_J^-(\vec\alpha).$$

If n is odd and $n+2k\equiv 1 \mod 4$ then

$$\tilde b^J_{n,k}(\vec\alpha)=i^{-k}2^{k-1}a_J^-(\vec\alpha),\qquad \tilde c^J_{n,k}(\vec\alpha)=i^{k}2^{k-1}a_J^+(\vec\alpha).$$

If n is odd and $n+2k\equiv 3 \mod 4$ then

$$\tilde b^J_{n,k}(\vec\alpha)=i^{k}2^{k-1}a_J^+(\vec\alpha),\qquad \tilde c^J_{n,k}(\vec\alpha)=i^{-k}2^{k-1}a_J^-(\vec\alpha).$$

Proof. By Lemma 7.3, it is sufficient to compare the initial conditions which are computed in Lemmas 6.2, 6.3, and 7.7. $\square$

8. Computation of the signatures

Lemma 8.1. *Let n and k be positive integers. The signature and the nullity of the link represented by the braid $\Delta^n\in B_{2k+1}$ are:*

$$\operatorname{Sign}\Delta^n=\begin{cases}-nk(k+1)+(-1)^{(n-1)/2} & \text{if } k\equiv n\equiv 1 \mod 2,\\ -nk(k+1) & \text{otherwise.}\end{cases}$$

$$\operatorname{Null}\Delta^n=\begin{cases}2k & \text{if } n\equiv 0 \mod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Apply [5]. $\square$

Proposition 8.2. *(a). Let n, k, J, and $\alpha_1,\dots,\alpha_J$ be positive integers such that $J\equiv n \mod 2$. Let $b=b^J_{n,k}(\alpha_1,\dots,\alpha_J)$ be as in Section 6. When $n\equiv 0 \mod 4$, we assume that $k=1$. Then we have ($\varepsilon_{n,k}$ is defined in (3))*

$$\operatorname{Null} b=\begin{cases}1 & \text{if } J\equiv(2k-1)n \mod 4 \text{ and } \alpha_1=\dots=\alpha_J=1,\\ 0 & \text{otherwise};\end{cases}$$

$$\begin{aligned}\operatorname{Sign} b-\operatorname{Null} b &= \operatorname{Sign}\Delta^n+(\alpha_1+\dots+\alpha_J)-J+(-1)^k\cdot\operatorname{Re} i^{\,n-1}\\ &=-nk(k+1)+(\alpha_1+\dots+\alpha_J)-J+\varepsilon_{n,k}.\end{aligned}$$

(b). Suppose that $n\equiv 2 \mod 4$ and $k\ge 1$. Let $b=b^2_{n,k}(\alpha_0,0)$ for $\alpha_0\ge 0$. Then Sign b *and* Null b *are computed by the formulas from the part (a) with $J=0$ and the term "$\alpha_1+\dots+\alpha_J$" (resp. the condition "$\alpha_1=\dots=\alpha_J=1$") replaced by "$\alpha_0$" (resp. by "$\alpha_0=1$").*

(c). Suppose that $n\equiv 0 \mod 4$ and $k=1$. Let $b=b^2_{n,k}(\alpha_0,0)$ for $\alpha_0\ge 0$. Then

$$\operatorname{Null} b=\begin{cases}2 & \text{if } \alpha_1=\dots=\alpha_J=1,\\ 1 & \text{otherwise};\end{cases}$$

Proposition 8.3. *(a). Let n, k, J, and $\alpha_1,\dots,\alpha_J$ be positive integers such that $J\equiv n \mod 2$, $n\not\equiv 0 \mod 4$, and $k\ge 2$. Let $c=c^J_{n,k}(\alpha_1,\dots,\alpha_J)$ be as in Section 6. Then we have ($\varepsilon'_{n,k}$ is defined in (4))*

$$\operatorname{Null} c=\begin{cases}1 & \text{if } J\equiv(2k+1)n \mod 4 \text{ and } \alpha_1=\dots=\alpha_J=1,\\ 0 & \text{otherwise};\end{cases}$$

$$\begin{aligned}\operatorname{Sign} c-\operatorname{Null} c &= \operatorname{Sign}\Delta^n+(\alpha_1+\dots+\alpha_J)-J-(-1)^k\cdot\operatorname{Re} i^{\,n-1}\\ &=-nk(k+1)+(\alpha_1+\dots+\alpha_J)-J+\varepsilon'_{n,k}.\end{aligned}$$

In particular, if n is even and $k>1$ then Null $c=$ Null b *and* Sign $c=$ Sign b *for $b=b^J_{n,k}(\alpha_1,\dots,\alpha_J)$.*

(b). Suppose that $n\equiv 2 \mod 4$ and $k\ge 2$. Let $c=c^2_{n,k}(\alpha_0,0)$ for $\alpha_0\ge 0$. Then Null $c=$ Null b *and* Sign $c=$ Sign b *for $b=b^2_{n,k}(\alpha_0,0)$*

Remark. Computations show that if $n\equiv 0 \mod 4$ and $k\ge 0$, then we have $\operatorname{Null} b=2(k-1)+\dots$, $\operatorname{Null} c=2(k-1)+\dots$, $\operatorname{Sign} b-\operatorname{Null} b=-2(k-1)+\dots$, and $\operatorname{Sign} c-\operatorname{Null} c=-2(k-1)+\dots$ where the dots stand for the corresponding expressions in Propositions 8.2 and 8.3. However, the method used in this paper is not sufficient to prove this fact.

Proof of Proposition 8.2. (The proof of Proposition 8.3 is analogous).

Case 1 (*n is odd*). We shall use the induction by J. Let us start with $J=1$.

We have $\tilde b^1_{n,k}(\alpha_1)=i^{-k}2^{k-1}a_1^-(\alpha_1)=(-2i)^k(1+\alpha_1)$ if $n+2k\equiv 1 \mod 4$ and

Table 1.

	$n+2k\equiv 1 \mod 4$	$n+2k\equiv 3 \mod 4$
α_1	$0\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$	$0\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$
$\operatorname{sign}\big(\tilde b^1_{n,k}(\alpha_1)/\tilde b^1_{n,k}(0)\big)$	$+\ +\ +\ +\ +\ +\ +\dots$	$+\ 0\ -\ -\ -\ -\ -\dots$
$\operatorname{Sign} b^1_{n,k}(\alpha_1)-\operatorname{Sign}\Delta^n$	$0\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$	$0\ 0\ 0\ 1\ 2\ 3\ 4\ \dots$
$\operatorname{Null} b^1_{n,k}(\alpha_1)$	$0\ 0\ 0\ 0\ 0\ 0\ 0\ \dots$	$0\ 1\ 0\ 0\ 0\ 0\ 0\ \dots$

Table 2.

	$n+2k\not\equiv J \mod 4$ and $\alpha_j=1$ for $j>1$	otherwise
α_1	$0\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$	$0\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$
$\operatorname{sign}\big(\tilde b(\alpha_1)/\tilde b(0)\big)$	$+\ 0\ -\ -\ -\ -\ -\dots$	$+\ -\ -\ -\ -\ -\ -\dots$
$\operatorname{Sign} b(\alpha_1)-\operatorname{Sign} b(0)+2$	$2\ 2\ 2\ 3\ 4\ 5\ 6\ \dots$	$2\ 1\ 2\ 3\ 4\ 5\ 6\ \dots$
$\operatorname{Null} b(\alpha_1)$	$0\ 1\ 0\ 0\ 0\ 0\ 0\ \dots$	$0\ 0\ 0\ 0\ 0\ 0\ 0\ \dots$

$\tilde b^1_{n,k}(\alpha_1)=i^k2^{k-1}a_1^+(\alpha_1)=(2i)^k(1-\alpha_1)$ if $n+2k\equiv 3 \mod 4$. Hence, by Lemma 3.1, the signatures and the nullities are as in Table 1 (we use that $b^1_{n,k}(0)\sim\Delta^n$). Thus, the statement of the lemma holds for $J=1$.

Now, suppose that we proved the statement of the lemma for smaller values of J. Let us fix posivive integers $\alpha_2,\dots,\alpha_J$ and denote $b(\alpha_1)=b^J_{n,k}(\alpha_1,\dots,\alpha_J)$, $\tilde b(\alpha_1)=\tilde b^J_{n,k}(\alpha_1,\dots,\alpha_J)$.

We have $b(0)=b^{J-2}_{n,k}(\alpha_J+\alpha_2,\alpha_3,\dots,\alpha_{J-1})$. By Corollary 7.9 and Lemma 7.7, the signs of the determinants are as in Table 2. Hence, by Lemma 3.1, the signature decrements and the nullities are as in the next two lines of Table 2.

It remains to note that when n is odd, $J\not\equiv n+2k$ iff $J\equiv(2k-1)n \mod 4$.

Case 2 ($n\equiv 2 \mod 4$). Similar to Case 1.

Case 3 ($n\equiv 0 \mod 4$). For $\alpha_2\neq 0$, we have $\det b^J_{n,k}(0,\alpha_2)=0$ and $\det b_{n,k}(1,\alpha_2)\neq 0$, hence, $\operatorname{Null} b^J_{n,k}(0,\alpha_2)=1$ and $\operatorname{Sign} b^J_{n,k}(0,\alpha_2)=\operatorname{Sign} b^J_{n,k}(1,\alpha_2)$ by Lemma 3.1. Since $\operatorname{Null} b^J_{n,k}(0,0)=2$, $\operatorname{Null} b^J_{n,k}(0,1)=1$, and $\operatorname{Null} b^J_{n,k}(1,1)=0$, we have $\operatorname{Sign} b^J_{n,k}(1,1)=\operatorname{Sign} b^J_{n,k}(0,1)=\operatorname{Sign} b^J_{n,k}(0,0)=\operatorname{Sign}\Delta^n$. The rest of the proof is similar to Case 1. $\square$

9. Proof of Theorem 1.1

We shall follow the scheme of the proof proposed in [7]. Let A be a curve as in Theorem 1.1. We associate to it a braid b (the construction from [7; Sections 3.4 – 3.5] with Δ^n instead of Δ). Then Murasugi-Tristram inequality imply (see [7] for details)

$$\operatorname{Null} b+1\geq|\operatorname{Sign} b|+m-e(b) \tag{19}$$

where $e(b)$ is the *exponent sum* of b, i.e. $e(b)=\sum k_j$ for $b=\prod\sigma_{i_j}^{k_j}$.

For an oval v, let D_v be the component of $\pi_n^{-1}(\pi_n(v))\setminus(\mathcal J\cup E_n)$ which contains v. Let $A'_{\rm odd}$ (resp. $A'_{\rm even}$) be the real (non-algebraic) smooth curve on $\mathbf R\Sigma_n$ which is obtained from $\mathbf RA$ by moving each odd (resp. even) oval v into the neighbouring

component of $D_v \setminus A_0$. Let $A''_{\rm odd}$ (resp. $A''_{\rm even}$) be the curve obtained from $A'_{\rm odd}$ (resp. from $A'_{\rm even}$) by moving each oval v into the component of $D_v \setminus A_0$ which is the nearest to $\mathcal{J}$ under the condition that the parity of v is not changed (see Figure 21). Let us denote the corresponding braids by $b'_{\rm odd}$, $b'_{\rm even}$, $b''_{\rm odd}$, and $b''_{\rm even}$ respectively.

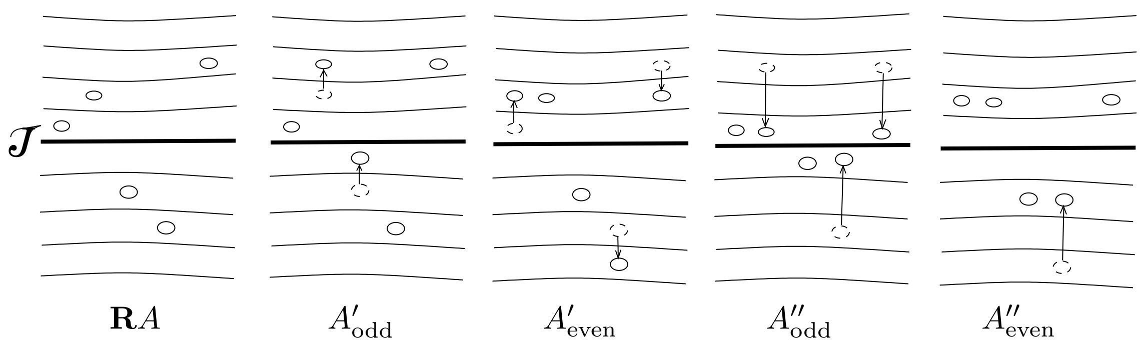


FIG. 21

Then we have $b''_{\rm odd} = b^J_{n,k}(\vec\alpha)$ and $b''_{\rm even} = c^J_{n,k}(\vec\alpha)$ for some $\vec\alpha = (\alpha_1, \dots, \alpha_J)$ where α_j is the number of ovals between two successive jumps over $\mathcal{J}$. In particular, we have $\alpha_1 + \dots + \alpha_J = \lambda$. Thus, the signatures of $b''_{\rm odd}$ and $b''_{\rm even}$ can be computed by Propositions 8.2 and 8.3. The condition $\lambda > J > 0$ ensures that

$$\operatorname{Null} b''_{\rm odd} = \operatorname{Null} b''_{\rm even} = 0. \tag{20}$$

When we pass from b to $b'_{\rm odd}$ (resp. to $b'_{\rm even}$), we $\lambda_{\rm odd}$ (resp. $\lambda_{\rm even}$) times performe the replacement $b_1\sigma_j^{-1}b_2 \longrightarrow b_1\tau\sigma_{j\pm1}^{-1}\tau^{-1}b_2$ where $\tau = \tau_{j,j\pm1} = \sigma_{j\pm1}^{-1}\sigma_j$. This is a composition of two band attachments: $b_1\sigma_j^{-1}b_2 \longrightarrow b_1b_2$ and $b_1b_2 \longrightarrow b_1\tau\sigma_{j\pm1}^{-1}\tau^{-1}b_2$ (it is clear that an inserting or a removing of $\sigma_j^{\pm1}$ is a band attachment). Hence, by Lemma 3.1(b), we have

$$\begin{aligned} |\operatorname{Sign} b'_{\rm odd} - \operatorname{Sign} b\,| + |\operatorname{Null} b'_{\rm odd} - \operatorname{Null} b\,| \le 2\lambda_{\rm odd}, \\ |\operatorname{Sign} b'_{\rm even} - \operatorname{Sign} b\,| + |\operatorname{Null} b'_{\rm even} - \operatorname{Null} b\,| \le 2\lambda_{\rm even}. \end{aligned} \tag{21}$$

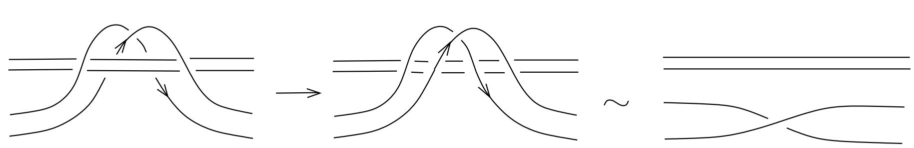


FIG. 22. PASSING FROM $b'_{\rm odd}$ TO $b''_{\rm odd}$

When we pass from $b''_{\rm odd}$ to $b'_{\rm odd}$ (resp. from $b''_{\rm even}$ to $b'_{\rm even}$), we perform several times the operation described in Lemma 5.1 (see Figure 22). Hence, by (20), we have

$$\begin{aligned} \operatorname{Sign} b'_{\rm odd} &= \operatorname{Sign} b''_{\rm odd}, \qquad \operatorname{Null} b'_{\rm odd} = 0, \\ \operatorname{Sign} b'_{\rm even} &= \operatorname{Sign} b''_{\rm even}, \qquad \operatorname{Null} b'_{\rm even} = 0. \end{aligned} \tag{22}$$

Theorem 1.1 is a direct comination of (19), (21), (22), Propositions 8.2 and 8.3, and the fact that $e(b) = -(\alpha_1 + \dots + \alpha_J) + e(\Delta^n) = -\lambda + nm(m-1)/2$.

Appendix A. Homogeneous skein systems and explicite formulas for the coefficients of $a^{\pm}_J$

Let us introduce the following notation: $\bar{J} = \{1, \dots, J\}$; $[J] = \{\{1,2\},\{2,3\},\dots,\{J-1,J\},\{J,1\}\}$; $[[J]] = \{A \subset [J] \mid \alpha \cap \beta = \varnothing \text{ for } \alpha,\beta \in A\}$; for $A \in [[J]]$, set $|A| = \bigcup_{\alpha\in A}\alpha$. It is clear that card $|A|$ is always even For $k \equiv J \mod 2$, let us denote $\{A \in [[J]] \mid \operatorname{card}|A| = J-k\}$ by $[[J]]_k$ and set

$$f_{J,k}(x_1,\dots,x_J) = \sum_{A\in[[J]]_k} x_A \qquad \text{where} \quad x_A = \prod_{j\in\bar{J}\setminus|A|} x_j\,.$$

It is clear that $f_{J,k}$ is a homogeneous polynomial of degree k.

Let us denote $\mathcal{F}_k = \{f_{J,k}\}_{J=2,4,6,\dots}$ for an even k and $\mathcal{F}_k = \{f_{J,k}\}_{J=1,3,5,\dots}$ for an odd k. So, we have

$$\begin{aligned}
\mathcal{F}_1 &= \{\, x_1,\ x_1+x_2+x_3,\ x_1+x_2+x_3+x_4+x_5,\ \dots\},\\
\mathcal{F}_2 &= \{\, x_1x_2,\ \ x_1x_2+x_2x_3+x_3x_4+x_4x_1,\\
&\qquad\quad (x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_6+x_6x_1)+(x_1x_4+x_2x_5+x_3x_6),\ \dots\},\\
\mathcal{F}_3 &= \{\,0,\ x_1x_2x_3,\ \ x_1x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_1+x_5x_1x_2,\\
&\qquad\ (x_1x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_6+x_5x_6x_7+x_6x_7x_1+x_7x_1x_2)+\\
&\qquad\ (x_1x_2x_5+x_2x_3x_6+x_3x_4x_7+x_4x_5x_1+x_5x_6x_2+x_6x_7x_3+x_7x_1x_4),\ \dots\},\\
&\ \vdots
\end{aligned}$$

On can easily check that each $\mathcal{F}_k$, $k \ge 1$, is a skein system of cyclic polynomials.

Proposition A.1. *If J is even then*

$$a^+_J(x_1,\dots,x_J) = \sum_{k=2,4,\dots,J} (2i)^k f_{J,k}(x_1,\dots,x_J) \quad \textit{and} \quad a^-_J = -4 - a^+_J.$$

If J is odd then

$$a^+_J(x_1,\dots,x_J) = 2 + \sum_{k=1,3,\dots,J} i(2i)^k f_{J,k}(x_1,\dots,x_J) \quad \textit{and} \quad a^-_J = 4 - a^+_J.$$

Proof. Follows from the fact that if $\{f_J\}$ is a skein system of cyclic polynomials then $f_J = c_0 + \sum_{k\equiv J(2)} c_k f_{J,k}$ where $c_0 = f_J(0,\dots,0)$ and c_k is the coefficient of $x_1x_2\dots x_k$ in the polynomial f_k.

Appendix B. Generalized skein relations

This appendix was written (as a section of the main body of the paper) at the moment when I knew already Theorem 4.2 but did not understand that it immediately implies Corollary 4.3. The identity (29) was used in the first version of the proof of Lemma 6.2 to treat the case of non-fiberable links. Now I do not know any application of these results but I decided to keep them just because they look nice.

In this appendix we work with links presented in the form of closed braids. A link $L \subset S^3$ is presented by a closed braid with m strings (m-braid) if $p|_L$ is a covering m of degree m where p is the projection $S^3 \setminus \ell = S^1 \times \mathbf{R}^2 \to S^1$ for some unknotted circle $\ell \subset S^3$. We always suppose the orientation of L to be induced by the projection $L \to S^1$. We use the language of braids just to siplify the notation. Everything can be reformulated for arbitrary link diagrams. We always assume that $B_k \subset B_m$ for $k < m$ identifying σ_j of B_k with σ_j of B_m.

Set $\delta_k = \sigma_1\sigma_2\dots\sigma_{k-1}$. Then $\Delta_k^2 = \delta_k^k$. The skein relation (11) can be reformulated as follows: $\Omega_b + (t - t^{-1})\Omega_{b\delta} - \Omega_{b\delta^2}$ for any $b \in B_m$ and $\delta = \delta_2 = \sigma_1$

Proposition B.2. *For any braid $b \in B_m$, $m \geq 3$, one has*

$$\Omega_b(t) + c_1\Omega_{b\delta}(t) + c_2\Omega_{b\delta^2}(t) + c_3\Omega_{b\delta^3}(t) + \Omega_{b\delta^4}(t) = 0 \tag{23}$$

where $\delta = \delta_3$, $c_1 = c_3 = -t^2 + 1 - t^{-2}$, $c_2 = -(t - t^{-1})^2$. In particular, for $t = i$,

$$\det b + 3\det(b\delta) + 4\det(b\delta^2) + 3\det(b\delta^3) + \det(b\delta^4) = 0. \tag{24}$$

Proof. Starting with a link diagram, one can construct a Seifert surface X using a standard algorithm based on so-called Seifert circles. Applying this algorithm to a link presented as a closed braid $b = \sigma_{i_1}^{\pm 1}\dots\sigma_{i_n}^{\pm 1}$ with m strings, one obtains m parallel equally oriented disks and n once-twisted ribbons where the j-th ribbon connects the i_j-th disk to $(i_j + 1)$-th one (see Fig. 23 for $b = \sigma_2\sigma_1^{-1}\sigma_2\sigma_2\sigma_1 \in B_3$).

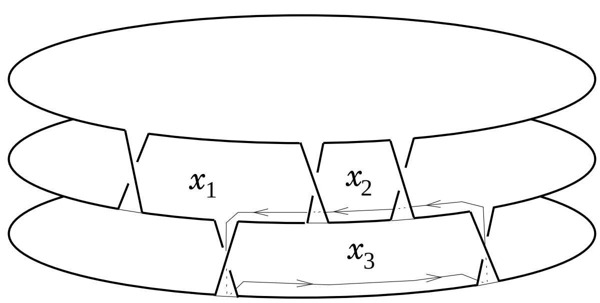
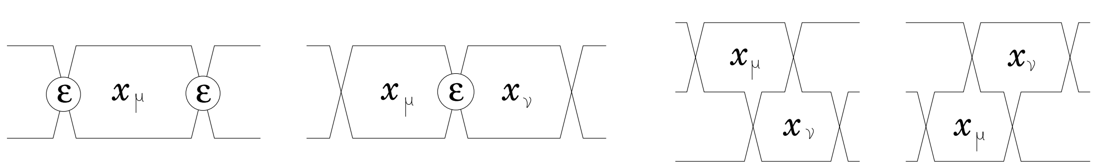


FIG. 23

As a base of $H_1(X)$ let us choose the cycles $x_1, \dots, x_s$, $s = n - m + 1$ which correspond to circuits in the positive direction around the bounded regions of the projection of b onto the plane (cycles x_1, x_2, x_3 in Fig. 23). Let $V = (v_{ij})$ be the corresponding Seifert matrix. All the mutual positions of x_μ and x_ν providing $v_{\mu\nu} \neq 0$ or $v_{\nu\mu} \neq 0$ are shown schematically in Fig. 24.



$v_{\mu\mu} = -\varepsilon$ $\quad$ $v_{\mu\nu} = (\varepsilon + 1)/2$, $v_{\nu\mu} = (\varepsilon - 1)/2$ $\quad$ $v_{\mu\nu} = 0$, $v_{\nu\mu} = 1$ $\quad$ $v_{\mu\nu} = -1$, $v_{\nu\mu} = 0$

FIG. 24

Aplying the above procedure to the braid $b_j := b\delta^j$ we obtain the $(s+2j)\times(s+2j)$-matrix V_j whose symmetrization $V_j^t = t^{-1}V_j - tV^T$ has the form

$$V_j^t = \begin{pmatrix} V_0^t & U & & & 0 \\ U^* & W & B & & \\ & B^* & A & \ddots & \\ & & \ddots & \ddots & B \\ 0 & & & B^* & A \end{pmatrix} \tag{25}$$

where $\quad A = \begin{pmatrix} t-t^{-1} & -t^{-1} \\ t & t-t^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} t^{-1} & 0 \\ -t & t^{-1} \end{pmatrix}, \quad B^* = \begin{pmatrix} -t & t^{-1} \\ 0 & -t \end{pmatrix},$

V_0^t is the $s\times s$ symmetrized Seifert matrix of $b = b_0$. Note, that U, U^*, and W are $s\times 2$-, $2\times s$-, and 2×2-matrices, common for all j. Let us prove that

$$\det V_0^t + c_1 \det V_1^t + c_2 \det V_2^t + c_3 \det V_3^t + \det V_4^t = 0 \tag{26}$$

for matrices V_j^t given by (25) where V_0^t, U, U^*, and W are arbitrary fixed $s\times s$-, $s\times 2$-, $2\times s$-, and 2×2-matrices (c_j are the same as in (23)).

Denote by $\widetilde{V}_j$ the $2j\times 2j$-matrix obtained from the right lower $2j\times 2j$-minor of V_j^t by replacing W with a 2×2-matrix $\widetilde{W}$ with indeterminate entries $\widetilde{w}_{\mu\nu}$. Then

$$\det \widetilde{V}_j = a_0^{(j)} + a_1^{(j)} \det \widetilde{W} + a_{11}^{(j)}\widetilde{w}_{11} + a_{12}^{(j)}\widetilde{w}_{12} + a_{21}^{(j)}\widetilde{w}_{21} + a_{22}^{(j)}\widetilde{w}_{22}$$

with $a_k^{(j)}, a_{\mu\nu}^{(j)} \in \mathbf{Z}[t,t^{-1}]$. The staight forward computation shows that

$$\begin{array}{lll} a_0^{(2)} = 1 & a_0^{(3)} = t^2 - 1 + t^{-2} & a_0^{(4)} = t^4 - t^2 - t^{-2} + t^{-4} \\ a_1^{(2)} = t^2 - 1 + t^{-2} & a_1^{(3)} = t^4 - t^2 - t^{-2} + t^{-4} & a_1^{(4)} = t^6 - t^4 + 1 - t^{-4} + t^{-6} \\ a_{12}^{(2)} = t^3 & a_{12}^{(3)} = t^5 - t^3 & a_{12}^{(4)} = t^7 - t^5 \\ a_{21}^{(2)} = -t^{-3} & a_{21}^{(3)} = t^{-3} - t^{-5} & a_{21}^{(4)} = t^{-5} - t^{-7} \end{array}$$

and $a_{11}^{(j)} = a_{22}^{(j)} = t^{-2}a_{12}^{(j)} + t^2 a_{21}^{(j)}$. It is easy to see that $\det V_j^t$ has form

$$\det V_j^t = a_0^{(j)} \det V_0^t + a_1^{(j)} \det V_1^t + a_{11}^{(j)} W_{22} + a_{12}^{(j)} W_{21} + a_{21}^{(j)} W_{12} + a_{22}^{(j)} W_{11} \tag{27}$$

where $W_{\mu\nu}$ is the determinant of the matrix obtained from V_1^t by deleting the row and the column contatining the (μ,ν)-entry of W.

Substituting the expressions (27) for $\det V_2^t, \dots, \det V_4^t$ into the left hand side of (26), we obtain a linear combination of $\det V_0^t$, $\det V_1^t$, and $W_{\mu\nu}$. A straight forward computation shows that all the coefficients vanish. □

Corollary B.3. *For any braid $b \in B_m$, $m \geq 3$, one has*

$$\Omega_b(t) + C_1\Omega_{b\Delta_3^2}(t) + C_2\Omega_{b\Delta_3^4}(t) + C_3\Omega_{b\Delta_3^6}(t) + \Omega_{b\Delta_3^8}(t) = 0 \tag{28}$$

where $C_1 = C_3 = -(t^3+t^{-3})^2$, $C_2 = 2(t^6+1+t^{-6})$. In particular, for $t = i$,

$$\det b - 2\det(b\Delta_3^4) + \det(b\Delta_3^8) = 0. \tag{29}$$

Proof. Let (23_j) be the result of substitution $b = \delta_3^j$ into (23). Then (28) is the sum of the identities $(23_0), \dots, (23_8)$ multiplied by $a_0, \dots, a_8$ respectively where $a_0 = a_8 = 1$, $a_1 = a_7 = t^2 - 1 + t^{-2}$, $a_2 = a_6 = t^4 - t^2 + 1 - t^{-2} + t^{-4}$, $a_3 = a_5 = -t^4 + t^2 - 2 + t^{-2} - t^{-4}$, $a_4 = -2(t^2 - 1 + t^{-2})$. (we use here $\Delta_3^2 = \delta_3^3$). □

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